



# Mathematics Teachers

# Circle

# around Problem Solving

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*By meeting regularly, a group of teachers from different schools promotes a culture of problem solving and engagement.*

Problem solving occupies a central place in NCTM's Standards (NCTM 2000). The process usually involves problems for which students do not know the solution method in advance and where new mathematics is learned. The Standards recommend that "students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking" (NCTM 2000, p. 52). Making problem solving a central part of teaching may be challenging to teachers who have limited experiences in learning and teaching mathematics in this way.

Math Teachers' Circles were developed with the aim of establishing a "culture of problem solving" among middle school mathematics teachers. This culture could then be carried back into these teachers' classrooms (American Institute of Mathematics, n.d.). This culture of problem solving is examined through the actions of one particular Math Teachers' Circle. Key features and their impact on particular teachers will also be discussed.

## **A BRIEF HISTORY**

Math Teachers' Circles originated in Bulgaria more than a hundred years ago and catered to the mathematical development of middle school and

**Fig. 1** Teachers think about and explore this problem while meeting in a Math Circle.

### Frogs and Toads

The frogs and toads want to swap places in the picture below. Find the minimum number of moves it would take, abiding by the following rules.



### THE RULES

1. A frog or toad can move into an empty space.
2. A frog or toad can hop over 1 frog or 1 toad to an empty space.
3. A frog or toad can only move forward.

In general, find the minimum number of moves for  $f$  frogs and  $t$  toads to swap places. (Assume that there is one blank space initially separating the groups.)

Source: Berlekamp, Conway, and Guy 1982; Bogomolny n.d.

high school students (Vandervelde 2009). Later, these Math Circles made their way to Russia, where communities sought to identify and encourage students who showed mathematical potential (Formin, Genkin, and Itenberg 1996). Over time, former members of these Math Circles who journeyed to the United States continued the tradition, having benefited from these circles when they were students.

The success of Math Teachers' Circles, or Circles, for short, aided by enthusiastic teachers, led to the creation of groups in northern California. The American Institute of Mathematics (AIM) started the first Circle in 2006 and has conducted regular workshops for teams interested in starting a Circle in their local area. Currently, there are twenty-two Circles, and new ones are being added every year (American Institute of Mathematics, n.d.).

### VIGNETTE OF A MATH TEACHERS' CIRCLE

The session begins with a group of five teachers and the facilitator having a friendly discussion over coffee and

bagels. The usual topics of discussion involve events at school and in the classroom. New members are usually introduced at this point, which helps foster camaraderie. As the teachers start taking their seats, the facilitator outlines the Frogs and Toads problem (see **fig. 1**) and provides an overview for the teachers.

The facilitator requests that teachers form groups, and he or she hands out empty boards, which will be used to track the position and movement of the frogs and toads, and different colored chips to represent each amphibian. One group comprises three teachers—Carol, Rita, and Jason. Initially this group explores possible ways that the frogs and toads could move in hopes of seeing a pattern.

Carol looks at the illustration of 3 frogs and 3 toads, denoted as (3, 3), and attempts to work out the number of moves by using 3 red chips and 3 yellow chips (representing 3 frogs and 3 toads). After a few unsuccessful attempts, Jason suggests that the group could try the process with a simpler case of (2, 2) or even (1, 1) and then work its way up.

The group members work out that 3 and 8 moves are needed for the (1, 1) and (2, 2) cases, respectively, and Carol immediately conjectures that the (3, 3) case would involve 13 moves. She bases this answer on her observation that the difference between 3 and 8 is 5 and assumes that the difference would remain constant for a growing number of frogs and toads. However, Jason remains skeptical and thinks that there might be something special about the numbers 3 and 8, but he is not sure what it is.

With the chips growing in number, the facilitator suggests that one group should follow the lead of the second group by tracking slides and jumps using the letter S (for slide) and J (for jump). The group works out the strings of Ss and Js for the (1, 1), (2, 2), and the (3, 3) cases (see **fig. 2**).

This group also notes that there are 15 moves for the (3, 3) case rather than 13, which was the initial conjecture. Further, they observe that the pattern is not alternate but symmetric on both sides of a "center." For example, in the (1, 1) case, the center

**Fig. 2** One group tracks its slides and jumps using capital letters.

(1, 1) case – SJS  
 (3 moves)  
 (2, 2) case – SJSJJSJS  
 (8 moves)  
 (3, 3) case – SJSJJSJJSJJSJS  
 (15 moves)

is J (second move); in the (2, 2) case, the center is JJ (fourth and fifth moves). Once again, on the basis of the (3, 3) case, Carol conjectures that the number of moves is growing by odd numbers— $3 + 5 = 8$ ,  $8 + 7 = 15$ . The next number would be  $15 + 9 = 24$  for the (4, 4) case. The group tests this conjecture and is excited to see that it is true. Jason suggests that the group should move to other cases containing a different number of frogs and toads, like (1, 2), and summarize their data in a table (see **table 1**).

The participants focus on the last column and try to observe a pattern in the number of moves as the number of frogs and toads increase. For example, they observe that the differences are 2, 3, 3, 4, 4, and 5, respectively. They use this information to conjecture that the (4, 5) case would be 29. However, they cannot seem to generalize this for  $f$  frogs and  $t$  toads. The facilitator, who is moving among the groups, points out that the eventual goal is to find the number of moves in terms of the number of frogs and toads. To move one group forward, the facilitator asks:

How would you determine the number of moves required for 100 frogs and 101 toads?

An interesting discussion of recursive and closed-form formulas ensues as Jason points out why Carol's method of looking at the differences would be limited when working with a large

number of frogs and toads. Jason then adds a column to their existing table to relate the number of frogs and toads to the number of moves (see **table 2**). However, the group still seems to focus on the differences (e.g.,  $[f + t + 4] - [f + t + 2] = 2$ ) between the rows. They are not observing that the numbers 1, 2, 4, 6, and so on in the last column could be represented as the product of  $f$  and  $t$ , with the general expression being  $f + t + ft$  for the total number of moves with  $f$  frogs and  $t$  toads.

Moving around the room and interacting with the groups, the facilitator observes that both groups have reached an impasse and decides that better progress may be made in a dis-

### 3 Ways to Cultivate a Successful Math Teachers' Circle

1. Choose nonroutine problems with various solution paths.
2. Form collegial groups with members who are not afraid to explore alternate solutions.
3. Elect a strong facilitator who keeps the task on track and moves it forward.

cussion involving all the teachers. The facilitator asks the groups to briefly relate their progress and then sketches this information on the board. While one group focuses on the difference in the total number of moves as they

**Table 1** A data table helps a group keep track of different frog-and-toad combinations.

Frogs	Toads	Squares	Moves
1	1	3	3
1	2	4	5
2	2	5	8
2	3	6	11
3	3	7	15
3	4	8	19
4	4	9	24

**Table 2** Another table allows teachers to look for recursive patterns and closed forms for predicting patterns.

Frogs	Toads	Squares	Moves	Pattern
1	1	3	3	$f + t + 1$
1	2	4	5	$f + t + 2$
2	2	5	8	$f + t + 4$
2	3	6	11	$f + t + 6$
3	3	7	15	$f + t + 9$
3	4	8	19	$f + t + 12$
4	4	9	24	$f + t + 16$

## Resources for Teachers' Circles

Berkeley Math Circle

- <http://mathcircle.berkeley.edu/>

Brazos Valley Math Teachers' Circle

- <http://www.math.tamu.edu/outreach/BVMTC/>

Charlotte Teacher's Circle

- <http://math.uncc.edu/index.php/organizations/charlotte-teachers-circle.html>

Map containing locations of operating and upcoming student and teachers' circles

- <http://www.mathteacherscircle.org/membercircles.html>

Mathematics Teachers Circle Network

- <http://www.theteacherscircle.org/>

National Association of Mathematics Circles

- [www.mathcircles.org](http://www.mathcircles.org)

New York City Mathematics Teacher's Circle

- <http://nymathcircle.org/teachers>

Tucson Math Circle

- <http://ime.math.arizona.edu/circles/teacher.html>

Utah Teachers' Math Circle

- <http://uuteacherscircles.wikispaces.com/>

progress down the rows, the other group attempts to generalize a pattern in the  $S$ s and  $J$ s. The facilitator elaborates on the recursive and closed-form formulas and reiterates with the entire group that the challenge is to find a closed-form solution.

At this point, Carol and Jason, still absorbed with the problem, observe that the numbers 1, 2, 4, 6, 9, 12, and 16 are just a product of the number of frogs and toads, respectively, and excitedly share this with the others. The facilitator further points out that  $f + t$  represents the number of slides and  $fz$ , the number of jumps, and that more jumps than slides occur as  $f$  and  $t$  grow. The final discussion impresses many of the teachers once they see how the various parts of the problem combine to produce the solution.

### KEY ASPECTS OF THE MATH TEACHERS' CIRCLE

This vignette shows some key patterns that are essential to the problem-

solving experience of the Circle. The first concerns the nature of the problems. All the problems are nonroutine, and the solution paths are unknown to the teachers at the outset. This scenario sets the stage for the teachers to explore and evaluate possible solution strategies. In some cases, appropriate tools for the exploration are provided, like the board and colored chips in the Frogs and Toads problem. During the exploration, teachers are drawn into making conjectures and justifying them to the group. For example, Carol makes conjectures about the number of moves in the different cases, and the entire group engages in verifying these conjectures.

It is important that the problem challenges the teachers at the appropriate level and engages their prior knowledge. In some cases, a brief introduction of a topic provides the support that is needed. By challenging the teachers at the appropriate level, excitement rather than frustration will result from their problem-solving attempts. Over time, this excitement will

contribute to the belief that problem solving, and mathematics in general, can be a fun and worthwhile endeavor. Last, the teachers have the opportunity to learn new mathematics, which is embedded in the tasks, as they engage in the problem-solving process. In this vignette, the teachers had an opportunity to find a general expression for the number of moves. In the process, they also learned the difference between recursive and closed-form formulas.

The second key aspect to the Circle is the establishment of a collegial and collaborative environment that fosters risk taking among the participants. This begins at the outset as the teachers and the facilitators interact informally over coffee. These interactions are crucial. It is important for new members to meet existing members and to feel comfortable sharing their thinking in the session, which, in turn, improves the interactions in the groups. The formation of groups is key in reducing the pressure on individual teachers to formulate the solution individually. More important, a great deal of learning occurs through interactions with other teachers as they try to (1) understand others' ideas, (2) verify a proposed conjecture, or (3) convince others of their way of thinking.

The exploratory component of the problems is important for several reasons. It allows for contributions from all the members as they assume different roles throughout the process. As well, a low level of risk taking will allow for potential contributions from all teachers in the group. For example, all teachers can participate in a game, record numbers, or make a conjecture (even if it turns out to be false). Toward the end as the group formulates a final solution, a feeling will emerge that the endeavor was collaborative. For example, Jason's contributions allow Carol to change her approach and begin with the (1, 1) case instead of the (3, 3) case. Progress

toward the solution is made collaboratively as teachers interact to generate and build on the common understanding of the problem. On many occasions, mathematicians and mathematics teacher educators are present during the Circle, and the teachers interact with them in groups.

Finally, the role of the facilitator is important to the working of the Circle to ensure that a continuous push for high-level thinking occurs among the teachers. The facilitator will usually allow the teachers to lead the process and intervene only during an impasse. This individual should emphasize collaboration among the teachers, encourage them to make and test conjectures, and provide hints when absolutely necessary. Most of the time the facilitator moves among the groups, listens to the discussions so as to understand the approaches being taken, and supports the teachers.

Since most of the facilitators are mathematicians, the teachers often get to see a different perspective on ways to think and solve mathematics problems. By internalizing thinking processes that they observe in the sessions, teachers can further develop their own problem-solving abilities. The facilitator also needs to choose proper problems and rise to the challenge. It is not unusual for a new facilitator to participate in a session before leading one. These future facilitators will usually join a group, work with the teachers, and observe the facilitators during the session. The existing Circles also maintain resources that can be used by a new facilitator (see the sidebar for website resources).

### THE IMPACT ON TEACHERS

Teachers attending the Circle have reported—through a questionnaire,

focus group, and personal interactions—the benefits for them and their teaching. The biggest benefit reported has been to their mathematical learning. One sixth-grade teacher made this comment:

I wanted to come here so that I would be challenged. I am not a mathematician. I am an educator, and I think I have a good understanding of what younger children need to carry them to the next level, but I am not real comfortable going to a real high level [in mathematics], and so it's very good for me. It's good for me to be put in that uncomfortable position, to know how my students are feeling when I put them in that uncomfortable position.

For some of the existing members of the Circle who were already engaging their students in problem-solving

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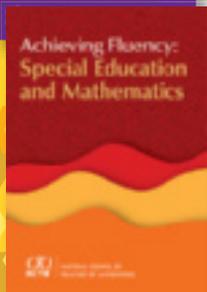
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activities, attending the Circle provided the support that they were not getting at school. A sixth-grade teacher reported that she felt validated by coming to the Circle meetings, which gave her the belief structure and courage to carry on the problem-solving focus that she had in her classroom.

Another major plus for teachers is attending a Circle to get ideas to inspire their own students. The Circle provides teachers with a continuous stream of good problems that can be adapted to various classroom scenarios. Further, the online resources also give them opportunities to see problems that are assigned in other Circles.

A seventh-grade teacher noted the following:

My goal in coming here is to find inspiration for myself to help me take math instruction beyond the walls of the classroom and beyond the limits of the textbook and to help give kids a sense of mathematics as a living discipline that engages their minds no matter where they are or what level they are. It's a process and an activity that can be fun and inspiring. You can shut out the whole rest of the world while you are engaged in a puzzle or a problem.

Teachers also have very specific ideas about their teaching. For example, one sixth-grade teacher shared with the group the following insight on the importance of allowing the students to cope with uncertainty as they do mathematics:

The whole idea of letting a group struggle with a problem and giving them time to do that is something that is not done in the schools very much because of the time. I think that is so powerful; it's just a great model for anybody involved in the classroom.

Another teacher pointed out how the interesting problems made his students excited, who in turn shared this excitement with their parents. Such experiences help parents understand a reform-oriented curriculum (NCTM 2000). Promoting students' excitement about mathematics seemed to be a major motivation for the teachers in coming to the Circle.

The problems and the collaborative experience of the Circle allowed the teachers to gain ideas for their own classrooms. One teacher discussed his concern, which was mirrored by others, about getting students to work constructively in groups. He pointed to the game that they played in pairs during a Circle session and thought that this would work in his classroom:

When you do that [play a game], you finally start getting kids to open up and to say things and to contribute something. Then you can expand upon that, like we did, and go into more difficult problems.

This teacher recognized the collaborative effort needed to involve all students in challenging problems:

We all contributed to solutions. I could not have solved these problems on my own, and yet I was able to be a contributor to solving the problems. We did come up with some solutions that we were working on through that. If I could do that here within this group, it gives me ideas, then I can see how other kids who may not get solutions on their own could help to contribute to solutions and finally be more active in their learning as far as the classroom is concerned.

One teacher reported on the change in her approach to teaching by even assigning problems to which she did not have the answer yet explored these jointly with the students. This

teacher also mentioned being confident to focus on the students' understanding the material rather than just "covering" the syllabus:

It's given me a little bit of freedom too because I know that it's good math going on. It has changed my teaching considerably or at least made me work toward that.

## DISCUSSION

By making problem solving the central focus of the Circles, the teachers are provided with opportunities to engage in nonroutine problems and get firsthand experience of the challenge and thrill of finding a solution. The formation of the collaborative groups and the role of the facilitator give the teachers a glimpse of what could be possible in their own classrooms. This is especially important in the current environment of high-stakes testing where the focus is on covering the content and preparing the students for tests.

The interaction with the mathematicians and mathematics teacher educators provides the teachers with a diverse experience in problem solving that potentially adds to their professional growth. Further, the interaction among the teachers themselves is a special highlight of the Circles as they get to interact with others who are also trying to implement problem solving in their classroom. The teachers learn about the challenges faced by their colleagues in other schools and the steps that they have taken to overcome these challenges. Gradually, over time, members develop a body of knowledge that is shared with the newer members through interactions with Math Teachers' Circles. With a continually growing number of Circles throughout the country, we hope that all teachers will acquire a culture of problem solving that they can pass on to their students.

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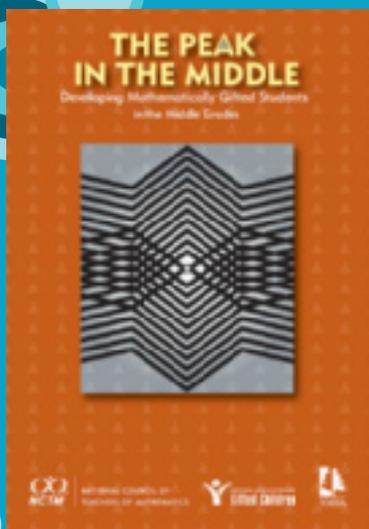
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(*Ed. note:* For additional information  
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"Patterns with Checkers" by Kathy  
J. Lewis in the March 2009 issue of  
*Mathematics Teaching in the Middle  
School.*)

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