Boomerang Fractions

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Note: These notes were compiled by Bob Klein based on a session led by Amanda. Bob owns all of the mistakes and credits Amanda for all of the genius. Amanda adds: "Boomerang Fractions should have credits including David Wilson, Gordon Hamilton, Joshua Zucker, and Richard Guy." BK adds: What shouldn't Richard Guy get credit for??

The boomerang fractions game is best illustrated by example, in this case, with a $\frac{1}{2}$ boomerang. Every Boomerang Fractions game begins with the number 1. You can either add the fraction to the previous number (in our case $\frac{1}{2}$), or you can take the reciprocal of the given fraction. Each game starts at 1 and the first step has to be adding the fraction.The goal is to get back to 1 as soon as possible and *if* possible. We can denote adding the fraction with $\rightarrow$ and taking the reciprocal with $\looparrowright$.

So the $\frac{1}{2}$ game looks like:

$$1 \rightarrow \frac{3}{2} \rightarrow 2 \looparrowright \frac{1}{2} \rightarrow 1.$$

We say that this game has *longevity* of 4 if there are a minimum of four steps ($\frac{3}{2}$, 2, $\frac{1}{2}$, 1) to get back to 1.

As another example, consider the $\frac{1}{3}$ game.

$$1 \rightarrow \frac{4}{3} \rightarrow \frac{5}{3} \rightarrow 2 \looparrowright \frac{1}{2} \rightarrow 1.$$

Convince yourself that there is no shorter sequence; that is, that the *longevity of* $\frac{1}{3}$ is 5.

Explore other fractions and their longevities. Are there fractions with infinite longevities? Longevities are *minimum* lengths. If you can't determine what are the *minimum* lengths (i.e. longevities), can you find an upper bound for the lengths of these sequences?

You could start with all unit fractions, that is fractions of the form $\frac{1}{n}$ where n is a natural number and $n \geq 1$ to see if there is a pattern or a formula for the longevity of $\frac{1}{n}$.

Explore.