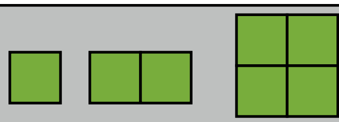
Number of Toonies	 **How to pay?**
1	
2	
3	
4	
5	
6	4 + 2
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	

Coins in Twoland

Nueva Math Circle, October 15, 2010
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In **Twoland**, the only coins are the toonies: 1, 2, 4, 8, 16, 32, 64, 128, and so on. The law says you must always pay with exact change; only the banks are allowed to make change.

In **Twoville** you must pay with zero, one, or two of each type of coin, never more than two. (If Twoville is too complicated for you, try Oneville where you can only pay with zero or one of each coin!)

For instance, to pay 6 toonies, you could pay with:

- one 4 and one 2 (and zero 1s)
- one 4, zero 2s, and two 1s
- or two 2s and two 1s.

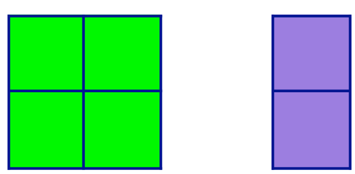
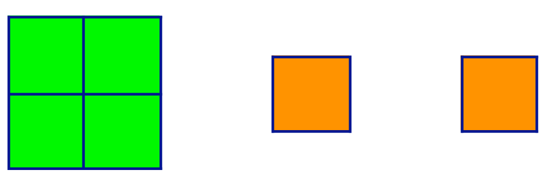
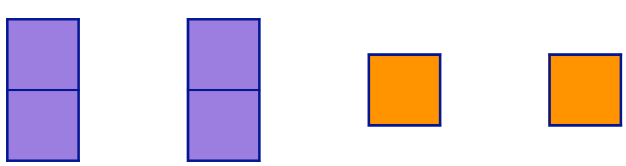
You could describe the three legal ways to pay by writing them in English, as I just did. You could write the list of legal ways to pay mathematically, too. For instance you might write

- 4 + 2
- 4 + 1 + 1
- 2 + 2 + 1 + 1.

Another, shorter way to write the list of legal ways to pay might be like this:

- 110
- 102
- 022 (or for short you might write just 22 here.)

You could also show these ways with some pictures:

- 
- 
- 

Now fill in the following table with the list of ways to legally pay each amount in Twoville, remembering that you cannot use more than two of each coin. What patterns do you find?

Number of Toonies	How many ways?	List of ways
1		
2		
3		
4		
5		
6	3	4+2, 4+1+1, 2+2+1+1
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		

Look at the table you created, describing possible payments of up to 16 toonies. Do you see any patterns in the number of ways to pay with exact change? How about in the list of all the different ways? Tell us about the patterns you find. How will the patterns work when you buy items that cost more than 16 toonies?

Look at the amounts that are one less than an exact single coin. Is there a pattern for the number of ways to pay 1, 3, 7, and 15? Why does that pattern work? How does that pattern continue?

What about the amounts that are exactly one single coin? Is there a pattern for the number of ways to pay 1, 2, 4, 8, and 16? Why does that pattern work? How does that pattern continue?

How about amounts that are one more than a single coin? Is there a pattern for the number of ways to pay 3, 5, 9, and 17? (Do you know how many ways there are to pay 17?) Why does that pattern work? How does that pattern continue?

Are there similar relationships for other lists of numbers? Why do they work?

Can you find a relationship between the number of ways to pay 11 toonies and the number of ways to pay 5? Do you see the same relationship between 13 and 6? By looking at the list of ways, can you find a reason for that relationship? Explain.

Can you find a relationship between the number of ways to pay 10 toonies and the number of ways to pay 4 or 5? Can you find a similar relationship between 12 and 5 or 6? By looking at the list of ways, can you find a reason for that relationship? Explain.

Can you find a relationship between the number of ways to pay 10 toonies and the number of ways to pay 12? Is there a similar relationship between 9 and 13? How about 8 and 14? Can you explain it by looking at the list of ways? As a hint, anything that costs at least 8 and less than 16 can be made using at most one 8, two 4s, two 2s, and two 1s.

In this chart, the number written under the box is the number of toonies you want to pay. Inside the box, you should write how many ways there are to pay that many toonies. For example, because 6 can be paid as 4 + 2 or 4 + 1 + 1 or 2 + 2 + 1 + 1, you should write a 3 in the box labeled 6. What patterns do you see?

1															
2	3														
4	5	6	7												
8	9	10	11	12	13	14	15								
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31

Number of Toonies	How Many Ways	List of Ways
1		
2		
3		
4		
5		
6	3	4 + 2, 4 + 1 + 1, 2 + 2 + 1 + 1
7		
8		
9		
10		
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Math Teachers' Circle
Coins in Twoland

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In Twoland, the only money is coins with value 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, and so on.

There is also a law that requires all purchases be made with exact change. Only the banks are allowed to make change.

Oneville

In Oneville, there is an additional law: purchases must be made with at most one copy of each coin. So for instance, paying 5 must be 4 + 1 (or 1 + 4, but we'll consider that the same). It would be illegal in Oneville to hand over 2 + 2 + 1 in payment.

1. How many ways are there to pay 1023?

2. How many ways are there to pay 1024?

3. How many ways are there to pay 1000?

Where do we live?

4. How many ways are there to write 349 in our system? Of course we can't write $2{\scriptstyle 14}9$ with a 14 in the tens place - why not? What are the laws?[1]

Twoville

Twoville, the capital of Twoland, turns out to be a much more interesting place. Their law requires that payments be made with at most two copies of each coin. So now you can pay 5 with 4+1 or 2+2+1, so there are two ways to do it.

5. How many ways are there to pay 1023?

6. How many ways are there to pay 1024?

7. How many ways are there to pay 1000?

8. How many different methods can we find for solving that last problem?

[1]We live in Tenland, in the town of Nineville.

Math Teachers' Circle
Coins in Twoland - Teacher Notes

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I think these problems can be used fairly directly with students. 1000 might be just a little bit large for them, but if you go too much smaller they will solve it with brute force instead of figuring out a general method that will let them work it out for any number.

Depending on their level, you might give them some worksheets that help them organize their efforts: a table of values with space to write a list of ways, for instance.

For some bigger hints, you might give them a table that's organized into rows for each power of two, so they write the number of ways to make 1 on the first line, 2 and 3 on the second line, 4, 5, 6, and 7 on the third line, and so on.

You might also suggest that they categorize the ways into groups depending on the number of 1s they use. Of course they'll quickly see that all the odd amounts require exactly one 1, and the even amounts require 0 or 2. Perhaps then they can discover the ways those amounts relate to previous cases.

I find that kids are often interested in understanding why our system of numbers actually works, with just one way to express each positive integer. Some kids may do better skipping over the Tenland problem, though, or exploring it differently. For example, ask about writing 399 as 40-1, or in other words 400 + -1. Most everyone will object, but you can point out how much easier it is to add 273 + 399 if you write it as 273 + 40-1 instead.

Naming the value of the 1-unit coin something that is humorous to your local clientele is also very useful. Here I left it unnamed. “Cents” is a bad idea because then it should be hundredths of something. I like “Zlotys” because they will think you are making it up, but maybe it's better to use something really made up.