Elgin Johnston Iowa State Univ.

Good "script" for NYMCP ?

Repeating Decimals

Warm up problems.

a. Find the decimal expressions for the fractions

$$\frac{1}{7}, \quad \frac{2}{7}, \quad \frac{3}{7}, \quad \frac{4}{7}, \quad \frac{5}{7}, \quad \frac{6}{7}.$$

What interesting things do you notice? Everybody asks one question.

b. Repeat the problem for the fractions

$$\frac{1}{13}, \quad \frac{2}{13}, \ldots, \frac{12}{13}.$$

What is interesting about these answers?

Every fraction has a decimal representation. These representations either terminate (e.g. $\frac{3}{8} = 0.375$) or or they do not terminate but are repeating (e.g.

$$\frac{3}{7} = 0.428571428571\ldots = 0.\overline{428571},)$$

where the bar over the six block set of digits indicates that that block repeats indefinitely.

1. Perform the by hand, long divisions to calculate the decimal representations for $\frac{1}{13}$ and $\frac{2}{13}$. Use these examples to help explain why these fractions have repeating decimals.

2. With these examples can you predict the "interesting things" that you observed in warm up problem b.? Look at the long division for $\frac{1}{7}$. Does this example confirm your reasoning?

3. It turns out that $\frac{1}{41} = 0.\overline{02439}$. Write out the long division that shows this and then find (without dividing) the four other fractions whose repeating part has these five digits in the same cyclic order.

4. As it turns out, if you divide 197 by 26, you get a quotient of 7 and a remainder of 15. How can you use this information to find the result when $2 \cdot 197 = 394$ is divided by 26? When $5 \cdot 197 = 985$ is divided by 26?

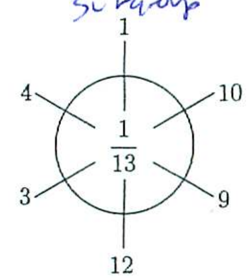
5. Suppose that when you divide N by D, the quotient is Q and the remainder is R. What are the possible answers when you divide $2N$ by D? When $3N$ is divided by D? When mN is divided by d?

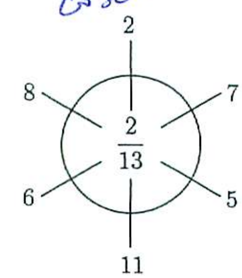
6. In warm up problem b. you found that

$$\frac{1}{13} = 0.\overline{076923} \qquad \text{and} \qquad \frac{2}{13} = 0.\overline{153846}.$$

The remainders encountered, in order, when performing the long divisions for these results are displayed in the *remainder wheels* below:

Subgroup — coset of multiplicative group of 1–12 mod 13.





How can you predict the remainders for the $\frac{2}{13}$ division from those for $\frac{1}{13}$ division? How does this situation relate to problems 4 and 5?

7. How can you use the remainder wheels above to find the decimal expansions for, say $\frac{4}{13}$ and $\frac{11}{13}$?

8. Explain why the remainders on the $\frac{1}{13}$ wheel are the remainders when the numbers 10^0, 10^1, 10^2, $10^3, \ldots$ are divided by 13. What similar statement can you make about the remainders on the other wheel?

9. Notice that if you take any two numbers from the $\frac{1}{13}$ wheel, take their product, and divide by 13, your remainder is another number on the wheel. However, this is not the case for the $\frac{2}{13}$ wheel. Why?

10. Referring back to Problem 3., write the remainder wheel for $\frac{1}{41}$. From this find the remainder wheels for $\frac{11}{41}$ and $\frac{20}{41}$. Use this remainder wheels to find the decimal expansions for $\frac{11}{41}$ and $\frac{20}{41}$. You can check your answers with a calculator.

11. Notice that in each of Problems 6. and 8., the remainder wheels produced have no numbers (e.g. remainders) in common. Why must this be the case? Given two remainder wheels for a given denominator, what can be said in general about the wheels?

12. Consider all of the possible remainder wheels for fractions with denominator 41. Will the number 0 appear in one of these remainder wheels? Why or why not? Will each of the numbers $1, 2, 3, \ldots, 40$ appear in a division wheel for 41? How do you know?

13. From what we have learned in Problems 9. and 10., how many different remainder wheels are there for 41?

Repeating Decimals—Notes

Warm-up problems. It is well known that the decimal expansions of the six fractions with denominator 7 can be obtained by cycling the digits of the repeating block. Is there anything of this sort happening with the fractions in part b?

1. At each step of the division process we do a subtraction to obtain a remainder between 1 and 12, then "bring down" 0 and divide the result by 13. Because there are only a finite number of these remainders possible, eventually a remainder must repeat and when that happens the division produces quotients and remainders identical to those produced before.
2. Write out the long division work for $\frac{1}{7}$ and $\frac{5}{7}$ and compare the two. Notice that at some point in the $\frac{1}{7}$ division we produce a remainder of 5. From this point on the process produces exactly the same results as those of the $\frac{5}{7}$ process. Thus the repeating block for $\frac{5}{7}$ has the same digits as those of the $\frac{1}{7}$ block. The digits occur in the same cyclic order, but the cycles start at different places because the divisions for the two fractions start at different remainders.
3. The other fractions can be identified by paying attention to the remainders as in the previous problem.
4. The given information about the division says that

   $$197 = 26 \cdot 7 + 15.$$

   Hence

   $$5 \cdot 197 = 5(26 \cdot 7 + 15) = 26(5 \cdot 7) + (5 \cdot 15).$$

   Division by 26 tells us how many "units" of size 26 can be pulled from the number. The display here says that we can take out $5 \cdot 7 = 35$ groups of 26 out plus any additional groups we can extract from $5 \cdot 15 = 75$. Because $75 = 2 \cdot 26 + 23$ we can pull two more 26s out of this part, and have 23 left over. Therefore when we divide $5 \cdot 197$ by 26 we get a quotient of $5 \cdot 7 + 2 = 37$ and a remainder of 23.
5. From the given information we know $N = D \cdot Q + R$, so

   $$mN = m(D \cdot Q + R) = D \cdot (mQ) + mR.$$

   How many "units" of size D can we pull out? We will get at least mQ from the initial term. Because $0 \leq mR \leq m(D-1)$, there could be anywhere between 0 and $m-1$ units of size D in mR. Thus when we divide mN by D, we will obtain a quotient

   $$mQ + k \text{ for some } 0 \leq k \leq m-1$$

   and some remainder r, which will be the remainder when mR is divided by D.

6. This relates to Problems 4 and 5. At any stage in the division you produce a remainder for the division to that point. When comparing the $\frac{1}{13}$ division with the $\frac{2}{13}$ division, we are doubling the dividend $(2 \cdot 1 = 2)$. In Problems 4 and 5 we saw how the remainder is affected when the dividend is multiplied by a positive integer.
7. With the wheels, you can recreate the digits of the quotient pretty easily. How?
8. If you stop the division process after, say 4 steps, the process you will have completed is the same as that you would do when dividing 10^4 by 13.
9. The key idea here is this:

   > suppose we have two integers, N_1 and N_2, and that when these numbers are divided by D the remainders are r_1 and r_2 respectively. Then when $N_1 \cdot N_2$ is divided by D the remainder will be the same as that when $r_1 \cdot r_2$ is divided by D.

   Now suppose we take two numbers from the $\frac{1}{13}$ wheel, say 9 and 12. The first is the remainder when 10^2 is divided by 13, and the second the remainder the 10^3 is divided by 5. Thus the product the remainder when $9 \cdot 12$ is divided by 13 is 4, which is the same as the remainder when $10^2 \cdot 10^3$ is divided by 13. But $10^2 \cdot 10^3 = 10^5$ and the remainder for this division will be the sixth number on the wheel, e.g. 4.

   This does not work with the $\frac{2}{13}$ wheel because the remainders here are those obtained when numbers of the form $2 \cdot 10^k$ are divided by 13. The product of two of these remainders will have the same remainder as a number $(2 \cdot 10^k)(2 \cdot 10^m) = 4 \cdot 10^{k+m}$. In particular this product does not give a number of the form $2 \cdot 10^n$, so we cannot expect the remainder to be in the wheel.

   (What is really going on here? The remainders for the $\frac{1}{13}$ wheel form a multiplicative group, which is actually a subgroup of the multiplicative group of integers $1, 2, 3, \cdots 12$. The elements of the other remainder wheel make a coset of this subgroup, but not a group.)
10. More practice with finding other remainder wheels given the on for $\frac{1}{41}$, and again Problems 4 and 5 are very useful.
11. If the same remainder appeared in different wheels, then when this point in the division is reached in each wheel, the results will be identical and we will be producing the same decimal digits and same subsequent cycle of remainders.
12. If we ever get a remainder of 0, then the division terminates, and this would mean we do not have a repeating decimal.
13. $40/5 = 8$.

In these problems we have worked with fractions of the form $\frac{k}{p}$ where p is an odd prime. The phenomena seen here will appear for any such fraction. Similar things happen for fractions of the form k/n where n is an odd integer not divisible by 5. The cycle, remainder wheel and group theory ideas still emerge, but only among fractions which are in lowest terms, e.g., with k relatively prime to n. This does not play out well for fractions $\frac{k}{n}$ if n is a multiple of 2 or 5.

This activity is based on the article "Fractions with Cycling Digit Patterns" by Dan Kalman. This paper appeared in *The College Mathematics Journal*, Vol. 27, No. 2, March 1996.

Repeating Decimals

Warm up problems.

a. Find the decimal expressions for the fractions

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What interesting things do you notice?

b. Repeat the problem for the fractions

$$\frac{1}{13},\ \frac{2}{13},\ldots,\frac{12}{13}.$$

What is interesting about these answers?

Every fraction has a decimal representation. These representations either terminate (e.g. $\frac{3}{8} = 0.375$) or or they do not terminate but are repeating (e.g.

$$\frac{3}{7} = 0.428571428571\ldots = 0.\overline{428571},)$$

where the bar over the six block set of digits indicates that that block repeats indefinitely.

1. Perform the by hand, long divisions to calculate the decimal representations for $\frac{1}{13}$ and $\frac{2}{13}$. Use these examples to help explain *why* these fractions have repeating decimals.

2. With these examples can you predict the "interesting things" that you observed in warm up problem b.? Look at the long divisions for $\frac{1}{7}$. Does this example confirm your reasoning?

3. It turns out that $\frac{1}{41} = 0.\overline{02439}$. Write out the long division that shows this and then find (without dividing) the four other fractions whose repeating part has these five digits in the same cyclic order.

4. Consider the repeating decimals

$$\frac{7}{13} = 0.\overline{538461} \qquad \text{and} \qquad \frac{5}{13} = 0.\overline{384615}. \tag{1}$$

Notice that the repeating block for the second decimal is obtained by taking the leading 5 from the first block and moving it to the end of the repeating block. Now repeat this process on the $\frac{5}{13}$ decimal to get the decimal $0.\overline{846153}$. What fraction has this decimal expansion? Repeat this process several more times and list the fractions that correspond to the decimals produced.

5. Another way to get from one decimal to the other in (1) is to multiply both side of the equation $\frac{7}{13} = 0.\overline{538461}$ by 10 to get

$$10 \cdot \frac{7}{13} = 10 \cdot 0.\overline{538461} = 5.\overline{384615}.$$

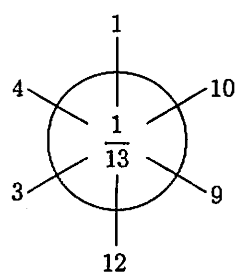
From this point how can you arrive at the second expansion in (1)?

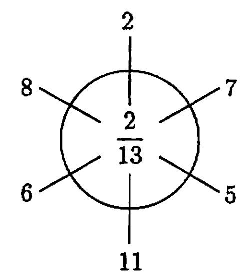
6. Note that when $70 = 7 \cdot 10$ is divided by 13, the remainder is 5. How does this information fit into the last problem? Formulate a rule for predicting the fraction obtained when the first digit of a repeating block is transferred to the end of the block.

7. In warm up problem b. you found that

$$\frac{1}{13} = 0.\overline{076923} \qquad \text{and} \qquad \frac{2}{13} = 0.\overline{153846}.$$

The remainders encountered, in order, when performing the long divisions for these results are displayed in the *remainder wheels* below:





For each wheel, how are the decimals for fractions $\frac{n}{13}$ with these numbers as numerators related? How does the order of the remainders (reading clockwise) relate to what you learned from problems 4, 5, 6? Formulate a rule for each wheel that takes you from one number to the next on the wheel in the clockwise direction?

8. Formulate a simple rule that can be used to generate the $\frac{2}{13}$ wheel from the $\frac{1}{13}$ wheel. What would the wheel for $\frac{5}{13}$ look like? What about $\frac{9}{13}$? Compare pairs of the wheels. If two of these wheels have a number in common the how many numbers do they have in common? Explain why this must be the case.

9. In problem 3. we noted that $\frac{1}{41} = 0.\overline{02439}$. Using the ideas developed above, write the remainder wheel for $\frac{1}{41}$. How many entries are on this wheel?

10. How many different remainder wheels are there for fractions with denominator 41? Use the ideas developed above to find all of these wheels. (Note, if one wheel is just a rotation of another wheel, then we will *not* consider them to be different.)

Repeating Decimals—Teacher's Notes

Warm-up problems. It is well known that the decimal expansions of the six fractions with denominator 7 can be obtained by cycling the digits of the repeating block. This is easier to see if you list the fractions and decimals in the order $\frac{1}{7}$, $\frac{3}{7}$, $\frac{2}{7}$, $\frac{6}{7}$, $\frac{4}{7}$, $\frac{5}{7}$. For the part b. warm up students should notice that the 12 decimals can be split into two sets; in each set, if the fractions are ordered in the right way, we will see that the digits in the repeating block again cycle. On the later pages of this document are the fractions for denominator 7, 13, and 21. The last is provided as an example of the cycling digit phenomenon with a non-prime denominator. Note that in this case we only consider the non-reducible fractions with denominator 21. One reason is that fractions like $\frac{26}{21} = \frac{2}{7}$ "belongs" to a different fraction set.

Some students might feel more confident with a calculator to help the through the individual long division calculations or the check that the repeating blocks in their decimal expansions are correct. However, I never allow a calculator with this activity. Rather, as students (working in groups) complete calculations for certain fractions, I ask them to come to the board and record their results. Students then compare board results with theirs and discuss the results until all are sure that the calculations are correct.

1. At each step of the division process we do a subtraction to obtain a remainder between 1 and 12, then "bring down" 0 and divide the result by 13. Because there are only a finite number of these remainders possible, eventually a remainder must repeat and when that happens the division produces quotients and remainders identical to those produced before.

   This reasoning can be applied to any fraction—there are only finitely many remainders possible in doing the long division to convert a fraction to a decimal. Because there are only finitely many possibilities, these remainders must repeat at some point. This will lead to a repeating block for the decimal. Note that if we ever get a remainder of 0, at some point in the division, then the decimal is terminating (for example $\frac{3}{8} = 0.375$.)

2. Write out the long division work for $\frac{1}{7}$ and $\frac{5}{7}$ and compare the two. Notice that at some point in the $\frac{1}{7}$ division we produce a remainder of 5. From this point on the process produces exactly the same results as those of the $\frac{5}{7}$ process. Thus the repeating block for $\frac{5}{7}$ has the same digits as those of the $\frac{1}{7}$ block. The digits occur in the same cyclic order, but the cycles start at different places because the divisions for the two fractions start at different remainders.

   Students should note that the remainders that appear in these divisions are the numerators in the family of fractions that belong to a cycle pattern for that fraction. On a later page is an illustration of the long division for $\frac{2}{13}$. The six remainders are in bold and these are the numerators of the fractions in the cycle group that contains $\frac{2}{13}$.

3. The other fractions can be identified by paying attention to the remainders as in the previous problem. There are only 5 remainders in this case and these remainders are the numerators of the five fractions with denominator 41 that are in a cycle family.

4. This exercise leads to a "natural ordering" of the fractions in a family. A fraction leads to the next fraction by taking the lead digit in a block and moving it to the back of the block.

5. The process that students should describe goes something like this:
$$\frac{7}{13} = 0.\overline{538461}.$$
Multiple by 10 to see
$$\frac{70}{13} = 5.\overline{384615}.$$
This can be written as
$$\frac{13 \cdot 5 + 5}{13} = 5 + \frac{5}{13} = 5.\overline{384615}.$$
Subtracting 5 leads to
$$\frac{5}{13} = 0.\overline{384615}.$$
The numerator 5 is the remainder when $70 = 10 \cdot 7$ is divided by 13.

6. Students should note that taking the leading digit of the repeating block and moving it to the end of the block is equivalent to multiplying the fraction (or repeating decimal) by 10, then dropping the integer part, that is, the part to the left of the decimal. When the analogous fraction is multiplied by 10 and the whole number part of the fraction dropped, the result is the fraction (less than one) that has repeating decimal expansion with block equal to the block produced earlier.

7. The next number, clockwise on the wheel, is found (as above) by multiplying the current number by 10, then finding the remainder on division by 13. Note that the two wheels have no numbers in common. This must be the case because the numbers on the wheel are the remainders obtained in doing the fraction to decimal conversion.

8. If these two remainder wheels shared a number, then the would have the same decimal expansions from these points, meaning they belong to the same cycle family. But $\frac{1}{13}$ and $\frac{2}{13}$ belong to different cycle families. Note that the $\frac{2}{13}$ wheel can be obtained from the $\frac{1}{13}$ wheel by multiplying each element in the latter wheel by 2, then taking the remainder on division by 13.

9. The process to move around the wheel in this case is: multiply by 10, then take the remainder on division by 41. This gives

$$1 \longrightarrow 10 \longrightarrow 18 \longrightarrow 16 \longrightarrow 37 \longrightarrow 1.$$

   Thus the fractions in the cycle family with $\frac{1}{41}$ are $\frac{10}{41}$, $\frac{18}{41}$, $\frac{16}{41}$, and $\frac{37}{41}$.

10. There are a total of 8 cycle families for the fractions $\frac{1}{41}, \frac{2}{41}, \ldots, \frac{40}{41}$. Each wheel has five spokes, and the integers at the ends of the spokes are the integers $1, 2, \ldots, 40$, each appearing exactly once. To generate the other wheels, find a digit n that does not appear in any wheel yet produced. Multiply the $\frac{1}{41}$ wheel by n and take the remainder on division by 41.

In these problems we have worked with fractions of the form $\frac{k}{p}$ where p is an odd prime. The phenomena seen here will appear for any such fraction. Similar things happen for fractions of the form k/n where n is an odd integer not divisible by 5. The cycle, remainder wheel and group theory ideas still emerge, but only among fractions which are in lowest terms, e.g., with k relatively prime to n. This does not play out well for fractions $\frac{k}{n}$ if n is a multiple of 2 or 5.

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$$\frac{1}{21} = 0.\overline{047619} \qquad \frac{11}{21} = 0.\overline{523809}$$

$$\frac{2}{21} = 0.\overline{095238} \qquad \frac{13}{21} = 0.\overline{619047}$$

$$\frac{4}{21} = 0.\overline{190476} \qquad \frac{16}{21} = 0.\overline{761904}$$

$$\frac{5}{21} = 0.\overline{238095} \qquad \frac{17}{21} = 0.\overline{809523}$$

$$\frac{8}{21} = 0.\overline{380952} \qquad \frac{19}{21} = 0.\overline{904761}$$

$$\frac{10}{21} = 0.\overline{476190} \qquad \frac{20}{21} = 0.\overline{952380}$$

$$\frac{1}{13} = 0.\overline{076923} \qquad \frac{7}{13} = 0.\overline{538461}$$

$$\frac{2}{13} = 0.\overline{153846} \qquad \frac{8}{13} = 0.\overline{615384}$$

$$\frac{3}{13} = 0.\overline{230769} \qquad \frac{9}{13} = 0.\overline{692307}$$

$$\frac{4}{13} = 0.\overline{307692} \qquad \frac{10}{13} = 0.\overline{769230}$$

$$\frac{5}{13} = 0.\overline{384615} \qquad \frac{11}{13} = 0.\overline{846153}$$

$$\frac{6}{13} = 0.\overline{461538} \qquad \frac{12}{13} = 0.\overline{923076}$$

$$\frac{1}{7} = 0.\overline{142857}, \quad \frac{2}{7} = 0.\overline{285714}, \quad \frac{3}{7} = 0.\overline{428571}, \quad \frac{4}{7} = 0.\overline{571428}, \quad \frac{5}{7} = 0.\overline{714285}, \quad \frac{6}{7} = 0.\overline{857142}.$$

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