

# Polydron Activity

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for the South East Ohio Math Teachers' Circle Summer Workshop

## (1) What is a polygon?

A polygon is a simple closed figure composed of straight line segments that lie in a plane.

## (2) What do you know about a polygon?

We name the polygon based on the number of sides. The fewest number of sides that a polygon can have is three. Some beginning names are triangle (3), quadrilateral (4), pentagon (5), hexagon (6), heptagon (7), octagon (8), nonagon (9), and decagon (10). The sum of the angles of an n-gon is fixed. You can use all of the diagonals from a fixed point to break the n-gon into  $n-2$  triangles and see that the sum of the angles of an n-gon will be  $(n-2)180^\circ$ .

For the previously mentioned polygons we will have angle sums of  $180^\circ$ ,  $360^\circ$ ,  $540^\circ$ ,  $720^\circ$ ,  $900^\circ$ ,  $1080^\circ$ ,  $1260^\circ$  and  $1440^\circ$ .

## (3) What is a regular polygon?

A regular polygon is a polygon in which all of the sides have the same length and all of the angles have the same degree measure.

## (4) What do you know about a regular polygon?

Based on the fact that the sum of the angles of an n-gon is fixed and each angle of a regular n-gon has the same measure, the size of each angle of a regular n-gon is also fixed. For example, a pentagon has an angle sum of  $540^\circ$ . Since each angle is the same size, each angle must be  $540^\circ/5$  or  $108^\circ$ . For a regular polygon, each angle will measure  $(n-2)180^\circ/n$ . For the regular n-gons mentioned above each angle will measure  $60^\circ$ ,  $90^\circ$ ,  $108^\circ$ ,  $120^\circ$ ,  $128\frac{4}{7}^\circ$ ,  $135^\circ$ ,  $140^\circ$  and  $144^\circ$ .

## (5) What is a tessellation?

A tessellation is created when a shape is repeated over and over again covering a plane without any gaps or overlaps.

## (6) Which regular polygons will tessellate and WHY?

In order for a polygon to tessellate the sum of the angles must add to  $360^\circ$ . The reason for this is that at any given point the sum of the angles at that point is  $360^\circ$ . The regular polygons will need to meet at a point and the sum of the angles at that point will need to be  $360^\circ$ . The only regular polygons that will tessellate are the triangle, square and hexagon. Can you explain why this is true?

Since the measure of each angle in a regular triangle will be  $60^\circ$  and  $360^\circ/60^\circ = 6$  we can see that six regular triangles will cover a point. For the square we will need four. If we look at the pentagon we see that  $360^\circ$  divided by  $108^\circ$  will not be a whole number. If we use three regular pentagons we will not have the point covered (since  $3 \times 108^\circ = 324^\circ$ ) and if we use four regular pentagons we will have overlap (since  $4 \times 108^\circ = 432^\circ$ ). Using this same approach, the only other regular polygon that will tessellate is the regular hexagon (since  $3 \times 120^\circ = 360^\circ$ ).

**(7) What is a platonic solid?**

A three dimensional figure whose faces are congruent regular polygons and where the same number of faces meet at every vertex.

**(8) How many platonic solids are there and WHY?**

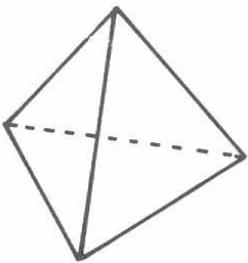
There are only FIVE platonic solids. We can see this by looking at the measure of the angles in regular n-gons. If we look at one fixed vertex of the platonic solid there must be the same number of regular polygons meeting at that vertex. If there are only two regular polygons at one vertex they would collapse against each other and there would not be a three dimensional figure. Therefore, we must have at least three regular polygons at one vertex. Furthermore, the sum of the angles at that vertex must be less than  $360^\circ$ , since a sum of  $360^\circ$  would mean the figures were laid flat and were not three dimensional .

For a regular triangle, each angle is  $60^\circ$ . If we have three triangles at one vertex we have what is known as the **Tetrahedron** because there are four faces. If we have four triangles at one vertex we have what is known as the **Octahedron** because there are eight faces. If we have five triangles at one vertex we have what is known as the **Icosahedron**. There are 20 faces in this figure. If we have six triangles at one vertex we have a total of  $360^\circ$  and the figure will be flat. Thus, using a regular triangle there are three different platonic solids, shown below.

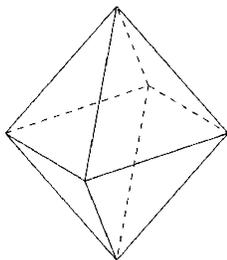
For a regular quadrilateral, also known as a square, the only possibility is having three squares meet at a vertex. Two is not enough and four will add to  $360^\circ$  and will be flat. When we have three squares meet at one vertex we have what is known as the **Hexahedron** because there are a total of 6 faces. This figure is shown below.

If we use a regular pentagon we can see that again the only possibility is having three of these shapes meeting at one vertex. If we have four the angles will add to more than  $360^\circ$ . This figure is known as the **Dodecahedron** because there are 12 faces. This figure is also shown below.

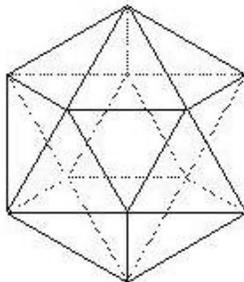
If we use a regular polygon with six or more sides we know that the angle size is  $120^\circ$  or more. We also know that we must have at least three of these polygons and the angles will add to  $360^\circ$  or more. Therefore, there are no platonic solids possible with regular polygons having more than five sides. Therefore, the only possible platonic solids are the ones shown below.



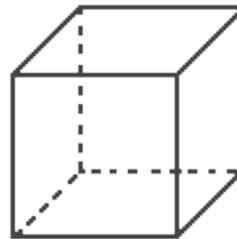
Tetrahedron



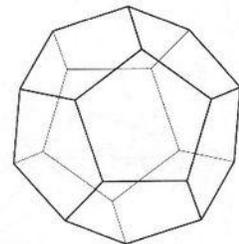
Octahedron



Icosahedron



Hexahedron



Dodecahedron

(9) Complete the following chart. What do you notice?

Platonic Solid	# of polygons at one vertex- M	# of vertices(sides) in one polygon- N	Total # of faces - F	Total #of vertices - V	Total # of edges - E
Tetrahedron					
Hexahedron					
Octahedron					
Dodecahedron					
Icosahedron					

We make note of a few facts. Since two faces meet in one edge we see that  $E = \frac{NF}{2}$ .

Since every vertex is shared by M faces we see that  $V = \frac{NF}{M}$

We also notice a pattern from the Table which is known as **Euler's Polyhedron Theorem**

$$F = 2 + E - V \text{ or } F + V = E + 2.$$

Using the above equations, we can see that M and N totally determine the remaining values for the platonic solid and thus for each configuration there is only one figure.

Using algebra, once we have M and N we can see that  $F = \frac{4M}{2M - MN + 2N}$

$$E = \frac{2MN}{2M - MN + 2N} \text{ and } V = \frac{4N}{2M - MN + 2N}.$$

Therefore, all values are uniquely determined by M and N.

These platonic solids have traditionally been related to the elements as follows:

Tetrahedron- Fire  
 Hexahedron-Earth  
 Octahedron – Air  
 Dodecahedron – Heaven  
 Icosahedron – Water.

(10) **Question for further exploration: What are the Archimedean Solids?**